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M. Awada
P. K. Townsend
M. Günaydin
G. Sierra

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M. Awada and P.K. Townsend

D.A.M.T.P., Silver Street
Cambridge, U.K.

M. Günaydin

Lawrence Livermore Laboratory, Livermore, CA 94556, U.S.A., and
Lawrence Berkeley Laboratory, Berkeley CA 94720, U.S.A.

G. Sierra

Univ. Complutense, Madrid, Spain.

Abstract

We show that the Riemannian manifold characterizing the scalar field interactions of d=9 Maxwell-Einstein (M-E) supergravity is the convex cone associated with a Jordan algebra of degree 2. This result is similar to that of a class of d=5, N=2, M-E supergravity theories associated with Jordan algebras of degree 3. We also construct the unique irreducible d=9 Yang-Mills supergravity which has non-compact gauge group $S\ell(2;R)$.

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Introduction.

In a recent work [1] three of us have shown that there is a 1-1 association of the members of a certain class of N=2, d=5 Maxwell Einstein (M-E) supergravity theories with unital, formally real, Jordan algebras of degree 3. (By Maxwell-Einstein supergravity we mean the coupling of an arbitrary number, n, of Abelian vector supermultiplets to supergravity). Dimensional reduction yields a class of N=2,d=4 M-E supergravity theories associated with these Jordan algebras (the general d=4 model has been discussed by de Wit et. al. [2]). A characteristic feature of these theories, in both d=5 and d=4, is that the Riemannian manifold $\mathcal{M}$ that is parametrized by the scalar fields, and which determines their non-polynomial interactions, is a symmetric space. In d=4 it has recently been shown that the only symmetric spaces allowed in M-E supergravity theory are either complex projective spaces or those associated with Jordan algebras [3], and the former can all be obtained as truncations of the latter.

In the light of these results for N=2 supergravity in d=5 and d=4 it is of interest to investigate whether N=4 Maxwell-Einstein supergravity theories have a similar interpretation in some spacetime dimensions. The maximal dimension for a M-E supergravity theory is 10, but there is only one scalar field in this case [4] so the scalar field manifold is rather trivial (it is SO(1,1)). The d=9 [5], d=4 [6], d=5 [7] and d=8 [8] maximal M-E supergravity theories have all been constructed recently and the scalar field manifolds determined. The results which agree, for d=4, with an earlier prediction [9] may be summarised by the formula

$$\mathcal{M}' = \frac{SO(10-d\,,\,n)}{SO(10-d) \times SO(n)} \tag{1.1}$$

for the manifold M' parametrized by the scalar fields of the Maxwell supermultiplets. The full Riemannian manifold M (parametrized by all the scalar fields) is locally $M' \times SO(1,1)$ (or $M' \times SU(1,1)/U(1)$ in d=4), the extra SO(1,1) (or SU(1,1)) factor coming from the scalar(s) in the graviton supermultiplet.

A geometrical interpretation of the d=9 result was presented in ref. (5). The purpose of the present work is to present an alternative interpretation in terms of the convex cones associated with Jordan algebras. This, we believe, has several advantages. Firstly it provides a link, through the theory of Jordan algebras, between N=2 and N=4 (or "maximal" in higher dimensions) M-E theories. Secondly, it provides a geometrical interpretation for the scalar field function that multiplies the kinetic term of the Maxwell fields. Thirdly, and most importantly, it provides an explicit construction of all the scalar field interactions in terms of a single quadratic polynomial, which is identifiable as the "norm form" of an associated Jordan algebra. Because the norm form is quadratic the Jordan algebra has degree 2. This is to be compared with the N=2, d=5 M-E theories associated with Jordan algebras of degree 3. The reduction in degree accounts for the relative simplicity of the d=9 M-E theories, compared to those of d=5.

As in the N=2 d=5 case the connection to Jordan algebras can be summarized by the result

$$M'(d=9) = \frac{Str_0(J)}{Aut(J)} \qquad (1.2)$$

for the d=9 scalar field manifold M', where Str_0 and Aut are the "reduced structure" and automorphism group, respectively, of the associated Jordan algebra J, here of degree 2. In agreement with ref. (5) these are the coset spaces $SO(n,1)/SO(n)$.

Dimensional reduction yields the Hermitian symmetric spaces [1]

$$\mathcal{M}'(d=8) = \frac{\mathrm{M\ddot{o}}(J)}{\widetilde{\mathrm{Str}}(J)} \tag{1.3}$$

in $d=8$, where $\mathrm{M\ddot{o}}(J)$ and $\mathrm{Str}(J)$ are the "Möbius" group and compact form of the structure group of J, respectively. For J of degree 2 these are just the Kählerian coset spaces $SO(n,2)/[SO(n) \times SO(2)]$, in agreement with the spaces obtained in $d=8$ by direct construction [8]. This shows incidently, that <u>all</u> of the $d=8$ M-E supergravity theories can be obtained by dimensional reduction from $d=9$.

In ref. (1) the scalar field manifold M was interpreted as a hypersurface of a larger space, the latter being identified as a convex cone. In the special case that M takes the form (1.2) this cone is the self-adjoint homogeneous convex cone (or "domain of positivity") of the Jordan algebra J. In the $d=9$ case the additional scalar field of the graviton supermultiplet serves as the "missing coordinate" of the cone which can therefore be directly identified with the scalar field manifold M. The manifold M' is the hypersurface analogous to that discussed in ref. (1) . The cone of the $d=9$ model is necessarily the domain of positivity of a Jordan algebra , now of degree two (i.e. this is not a special case as in $d=5$)

In the following section we present a more detailed discussion of convex cones, their relation to Jordan algebras, and a discussion of their relevance to $d=5$ and $d=9$ supergravity. We hope to persuade the reader that the mathematics of convex cones provides a unified understanding of M-E theories in both $d=5$ and $d=9$.

In the subsequent section we present an explicit construction of the $d=9$ M-E supergravity theory. This differs from the previous construction

[5] in two respects. Firstly we construct the theory in the form that is dual to that of ref. (5) in that we replace the second rank antisymmetric tensor gauge potential $A_{\mu\nu}$ by the dual $A_{\mu\nu\rho\sigma\lambda}$. This has the advantage that "modified" field strengths [10] are avoided. Secondly we show how the notation may be simplified to accord with our observation that the full space M parametrized by the scalar fields is the convex cone of a Jordan algebra of degree 2.

Like the d=5 N=2 "magical" supergravity theories discussed in ref. (1) the d=9 M-E theories are irreducible in that there is a symmetry linking every field of the model. It is a combination of supersymmetry and the non- compact SO(n,1) symmetry. An interesting question is whether a subgroup of SO(n,1) can be gauged in such a way that this irreducibility is not lost. As the (n + 1) rep. branches into the n + singlet rep. of the compact subgroup SO(n) it is obvious that we shall need to gauge a non-compact subgroup of SO(n,1) of dimension n for which the n + 1 rep. of the latter is the adjoint rep. The only simple gauge group that fits the requirements is $S\ell(2;R) \cong SO(2,1)$ for n = 2, which was mentioned as a possibility in ref. (5). We construct the $S\ell(2;R)$ Yang-Mills-Einstein supergravity explicitly and show that the gauging does not induce a scalar potential term in the action. This is in all respects analogous to the gauging of SU(3,1) in the N = 2 d = 5 case [11].

As previously mentioned one can dimensionally reduce the results for d=9 to obtain M-E or Y-M-Einstein supergravity theories in d < 9, in particular those in d=8. In conclusion we present a brief discussion of these lower dimensional models.

2. Convex Cones.

A convex cone is an open subset of points in an (n+1)-dimensional Euclidean space having certain properties. Every convex cone has a certain "characteristic function" ω [12-13] which is a homogeneous function of degree n+1 in the cartesian coordinates of the points of the cone, and which can be used as a kernel for a positive definite, symmetric, "natural" metric $g = -d^2 \ln\omega$. Furthermore, it can be shown that ω^2 is a polynomial [12]. In order to get directly to the results of interest here we shall take the definition of a convex cone to be a Riemannian space whose metric is of this form . The polynomial ω^2 is generally factorizable as

$$\omega^2(\{\xi\}) = \mathcal{N}_1^{n_1}(\{\xi_1\})\,\mathcal{N}_2^{n_2}(\{\xi_2\})\,\ldots\,\mathcal{N}_k^{n_k}(\{\xi_k\}), \tag{2.1}$$

i.e. into powers of polynomials $\mathcal{N}_i$ depending on a subset $\{\xi_i\}$ of the coordinates $\{\xi\}$, the subsets being disjoint. In this case the cone C has a metric that is a direct sum of k factors. Thus it will be sufficient to consider characteristic functions of the form

$$\omega^2(\xi) = \left[\mathcal{N}(\xi)\right]^{\frac{2(n+1)}{\nu}} \tag{2.2}$$

where $\mathcal{N}$ is a homogeneous polynomial of degree ν. In this case the metric $-d^2\ln\mathcal{N}$ defined in terms of the polynomial $\mathcal{N}$ differs from the natural metric $-d^2\ln\,\omega$ only by an unimportant constant factor. We shall choose this factor such that the metric is $a = -\nu^{-1}d^2\ln\mathcal{N}$. On the coordinate basis $\{e_I\}$ for which

$$\xi = \xi^I e_I \,, \tag{2.3}$$

this metric takes the form

$$a_{IJ} = -\frac{1}{\nu}\partial_{IJ}\ln \mathcal{N}(\xi), \tag{2.4}$$

(where $\partial_{IJ} = \partial_I\partial_J$ and $\partial_I = \partial/\partial\xi^I$). The polynomial $\mathcal{N}$ can be interpreted in certain cases (to be specified later) as the <u>norm form</u> of a unital, formally real, Jordan algebra, whose elements are $\{\vec{\xi}\}$. The degree, ν of homogeneity of $\mathcal{N}$ is also the degree of the algebra.

Notice that the metric a becomes singular at $\mathcal{N} = 0$, but the cone consists of a connected set of points for which $\mathcal{N} > 0$ so the points on the boundary $\mathcal{N} = 0$ are <u>not</u> part of the cone. (What is often referred to as the lightcone of Minkowski space in special relativity is a actually a lightfront, and is not a cone). The cone C is foliated by the hypersurfaces of constant $\ln\mathcal{N}$ (which is real as $\mathcal{N} > 0$). The normal to the hypersurface $\ln\mathcal{N}$ = constant is denoted by n and has components

$$n_I = \frac{1}{\nu}\partial_I \ln\mathcal{N} \tag{2.5}$$

on the dual basis e^I;

$$e^I = \delta^{IJ} e_J \quad . \tag{2.6}$$

Now as a consequence of the homogeneity of $\mathcal{N}$ we have

$$\xi^I n_I = 1 \tag{2.7}$$

and differentiating w.r.t. ξ^I we obtain

$$n_I = a_{IJ}\,\xi^J . \tag{2.8}$$

Because of the form (2.4) taken by a_{IJ} the Christoffel connection reduces to

$$\begin{aligned}\Gamma_{IJK} &= \tfrac{1}{2}\, a_{IJ,K} \\ &= -\frac{1}{2\nu}\,\partial_{IJK} \ln \mathcal{N} ,\end{aligned} \tag{2.9}$$

and as a consequence of the homogeneity of $\mathcal{N}$,

$$\Gamma_{IJ}{}^{K}\, n_K = -\, a_{IJ} \tag{2.10}$$

The Riemann tensor of C takes the form

$$R_{IJK}{}^{L} = 2\,\Gamma_{K[I}{}^{M}\,\Gamma_{J]M}{}^{L} \tag{2.11}$$

which is valid in those coordinate systems in which the metric has the form (2.4). It follows from (2.10) that

$$R_{IJK}{}^{L}\, n_L = 0 \tag{2.12}$$

which states that C has non-vanishing sectional curvature only for sections tangent to the hypersurfaces $\ln \mathcal{N}$ =const. This is a characteristic feature of a cone.

On a given hypersurface we can set up coordinates ϕ_x ,$x = 1,2...n$. The n tangent vectors to the hypersurface t_x have components

$$t^I{}_x = -\left(\frac{\nu}{2}\right)^{\frac{1}{2}} \xi^I{}_{,x} \tag{2.13}$$

where the factor is inserted for later convenience. The orthogonality of the tangent vectors to the normal is expressed by the relation

$$n_I t^I{}_x = 0 \tag{2.14}$$

The metric induced on the hypersurface by a is

$$g_{xy} = a_{IJ} t^I_x t^J_y \tag{2.15}$$

At each point C we can introduce a new basis of vectors,

$$\left\{ n_I \, , \; t_{Ix} \equiv a_{IJ} t^J{}_x \right\}$$

The metric a has the expansion

$$a_{IJ} = n_I n_J + t_{Ix} t_{Jy} g^{xy} \tag{2.16}$$

on this basis, the coefficients being fixed by (2.8) and (2.15). Similarly the Christoffel connection has the expansion

$$\Gamma_{IJK} = n_I n_J n_K - 3\, n_{(I}\, a_{JK)} - T_{xyz}\, t^x_I\, t^y_J\, t^z_K\, , \qquad (2.17)$$

which introduces the third rank symmetric tensor T_{xyz}, defined on the hypersurface. The coefficients of the first two terms in this expression are fixed by (2.10)

Notice that as a consequence of (2.13)

$$\frac{\partial}{\partial\phi^x} = -\left(\frac{2}{\nu}\right)^{\frac{1}{2}} t^I_x\, \partial_I \qquad (2.18)$$

so that

$$a_{IJ,x} = -\left(\frac{2}{\nu}\right)^{\frac{1}{2}} t^K_x\, a_{IJ,K} = -2\left(\frac{2}{\nu}\right)^{\frac{1}{2}} t^K_x\, \Gamma_{IJK}\, . \qquad (2.19)$$

But $a_{IJ,x}$ can also be expressed as

$$a_{IJ,x} = 2\, n_{(I}\, n_{J),x} + 2\, t_{(Iy}\, t_{J)z;x}\, g^{yz} \qquad (2.20)$$

Now it is straightforward to show that

$$n_{I,x} = \left(\frac{2}{\nu}\right)^{\frac{1}{2}} t_{Ix} \qquad (2.21)$$

so that by comparing (2.19) with (2.20) and using (2.17) we deduce that

$$t_{Ix;y} = \left(\frac{2}{\nu}\right)^{\frac{1}{2}}\left(n_I\, g_{xy} + T_{xyz}\, t^z_I\right) \qquad (2.22)$$

Similarly, we have

$$t^I{}_{x;y} = -\left(\frac{2}{\nu}\right)^{\frac{1}{2}}\left(\xi^I g_{xy} + T_{xyz}\, t^{Iz}\right). \tag{2.23}$$

Now from (2.17) we have that

$$T_{xyz} = -t^I_x\, t^J_y\, t^K_z\, \Gamma_{IJK}, \tag{2.24}$$

as the definition of T_{xyz}. Hence

$$\begin{aligned} T_{xyz;w} = &- 3\, t^I_{(x}\, t^J_y\, t^K{}_{z);w}\, \Gamma_{IJK} \\ &+ \left(\frac{2}{\nu}\right)^{\frac{1}{2}} t^I_x\, t^J_y\, t^K_z\, t^L_w\, \Gamma_{IJK,L} \end{aligned} \tag{2.25}$$

(by using (2.18) in the last term). The first term may be simplified by means of (2.12). To simplify the last term we make use of the expression (2.9) for Γ_{IJK} in terms of . This may be written as

$$\Gamma_{IJK} = -\frac{1}{2\nu}\left(\frac{\partial_{IJK}\mathcal{N}}{\mathcal{N}}\right) - \frac{3\nu}{2}\, n_{(I}\, a_{JK)} + \frac{\nu^2}{2}\, n_I\, n_J\, n_K, \tag{2.26}$$

from which follows

$$\begin{aligned} \Gamma_{IJK,L} = &-\frac{1}{2\nu}\left(\frac{\partial_{IJKL}\mathcal{N}}{\mathcal{N}}\right) + 4\nu\, n_{(I}\, \Gamma_{JKL)} - 3\nu^2\, a_{(IJ}\, n_K\, n_{L)} \\ &+ \frac{\nu^3}{2}\, n_I\, n_J\, n_K\, n_L + \frac{3\nu}{2}\, a_{(IJ}\, a_{KL)}. \end{aligned} \tag{2.27}$$

Because of the orthogonality of n and t_x only the first and last terms in

(2.27) contribute in (2.25). In this way we arrive at the result

$$T_{xyz;w} = \left(\frac{2}{\nu}\right)^{\frac{1}{2}}\left[\frac{3}{2}(\nu-2)\, g_{(xy}\, g_{z)w} - 3\, T_{(xy}{}^{v}\, T_{z)wv} - \frac{1}{2\nu}\, t^{I}_{x} t^{J}_{y} t^{K}_{z} t^{L}_{w} \left(\frac{\partial_{IJKL}\mathcal{N}}{\mathcal{N}}\right)\right]. \tag{2.28}$$

Notice that $T_{xyz;\omega}$ is <u>totally symmetric</u> in $xyz\omega$. This observation is needed to determine the integrability conditions of (2.22) and (2.23). From (2.22) we have

$$t_{Ix;y;z} = \left(\frac{2}{\nu}\right)^{\frac{1}{2}}\left[n_{I,z}\, g_{xy} + T_{xyu;z}\, t^{u}_{I} + T_{xyu}\, t^{u}_{I\,;z}\right]. \tag{2.29}$$

Hence,

$$t_{Ix;[y;z]} = \frac{2}{\nu}\left[t_{I[z}\, g_{y]x} + T_{ux[y}\left(\delta^{u}_{z]}\, n_{I} + T_{z]}{}^{u}{}_{v}\, t^{v}_{I}\right)\right], \tag{2.30}$$

where we have used (2.21) and (2.22). The first term in the round bracket vanishes because of the symmetry of T_{xyz}, leaving

$$t_{Ix;[y;z]} = \frac{2}{\nu}\left[g_{x[y}\, g_{z]v} + T^{u}{}_{x[y}\, T_{z]uv}\right] t^{v}_{I} \tag{2.31}$$

But the left-hand-side is proportional to the Riemann tensor K of the hypersurface $\ln\mathcal{N}$ = const. Thus we deduce that

$$K_{yzxv} = -\frac{4}{\nu}\left(g_{x[y}\, g_{z]v} + T^{u}{}_{x[y}\, T_{z]uv}\right) \tag{2.32}$$

This is just the curvature of C projected onto the hypersurface, so the hypersurface has vanishing second fundamental form. It is not difficult to demonstrate this directly.

It turns out that the cases of interest for supergravity are those for which ν is 1, 2, or 3, for which the last term in (2.28) vanishes. We shall consider each case in turn, starting with $\nu = 3$. In this case the polynomial $\mathcal{N}$ can be written as

$$\mathcal{N} = C_{IJK}\,\xi^I \xi^J \xi^K \tag{2.33}$$

and (2.28) becomes

$$T_{xyz;w} = \sqrt{\tfrac{3}{2}}\left[g_{(xy}\, g_{z)w} - 2\, T_{(xy}{}^{u}\, T_{z)wu} \right] \tag{2.34}$$

This is identical to the equation (2.17) of the first of refs. (1) and is the basic equation of the model because its integrability condition is again (2.32) and so it determines the curvature of the hypersurface $\ln \mathcal{N}$ = const. on which T_{xyz} is defined. This hypersurface can be identified with the manifold M parametrized by the scalar fields of d=5, N=2, M-E supergravity. In the case that $T_{xyz;\omega} = 0$ M is evidently a symmetric space and, in fact, is the space $Str_0(J)/Aut(J)$ associated with a unital, formally real, Jordan algebra J of degree three. The association comes about because, in this case, the tangent space components T_{ijk} of the T-tensor are constants that can be identified as the non-trivial structure constants of the algebra (i.e. those not involving the unit element of the algebra). The polynomial $\mathcal{N}$ is identifiable as the "norm form" of the algebra, which is of degree 3 because $\mathcal{N}$ is cubic [1].

Now we turn to the case of $\nu = 2$. We can write

$$\mathcal{N} = C_{IJ}\xi^I \xi^J \tag{2.35}$$

in this case. From (2.24) and (2.26) we have

$$T_{xyz} = \frac{1}{2\nu} t^I_x t^J_y t^K_z \left(\frac{\partial_{IJK}\mathcal{N}}{\mathcal{N}} \right) = 0 \tag{2.36}$$

because $\mathcal{N}$ is quadratic. This is consistent with (2.28) for $\nu = 2$ and the curvature tensor K reduces to

$$K_{yzxv} = -2\, g_{x[y}\, g_{z]v} , \tag{2.37}$$

so the hypersurfaces $\ln \mathcal{N}$ = constant of C are maximallly symmetric spaces. Positivity of a requires that $\mathcal{N}$ have "Minkowski signature", i.e. (+,−,−,...−). hence $\mathcal{N}$ is an SO(n,1) invariant. But symmetries of are isometries of a so the hypersurface has isometry group SO(n,1). As it is a maximally symmetric space it must be SO(n,1)/SO(n).

One can also arrive at this conclusion by noticing that as the T-tensor vanishes the associated Jordan algebra has **vanishing** non-trivial structure constants. Such algebras are sub-algebras of Clifford algebras of quadratic forms and are denoted J(Q). The reality of J requires that $\mathcal{N}$=Q have Minkowski signature, and the associated convex cone (which is the set of points in exp(J)) is locally SO(1,1)×SO(n,1)/SO(n). The latter factor is just $Str_0(J)/Aut(J)$ for J of degree two. and this space can be identified with the scalar field manifold M′ of d = 9 M-E supergravity.

Finally, for completeness, we may consider the case $\nu = 1$. In this case

$$\mathcal{N} = \xi \tag{2.38}$$

with ξ a real number. The cone C is necessarily one-dimensional and is just the positve real axis. The hypersurface $\mathcal{N}$ = constant is just one point so the equations (2.28) and (2.32) are not applicable. The associated Jordan algebra is simply the algebra of the real numbers. This case is relevant to d=10 M-E supergravity as the manifold parametrized by the single scalar field of this model can be identified with the one- dimensional cone.

3. $\underline{d} = 9$ M-E supergravity.

We present here the $d = 9$ M-E supergravity in the dual form to that given in ref. (5). This could be obtained by a duality transformation, following the procedure of ref. (10), but we prefer to obtain it directly, in order to have it in our notation and conventions. After presenting the Lagrangian in this form we show how it may be rewritten in a simpler form by making use of the connection with the geometry of convex cones.

Our conventions are as follows. We use the "mostly minus" metric, $\eta = (+1,-1,-1,\ldots-1)$ for the $d = 9$ spacetime tangent space. The charge conjugation matrix C is symmetric and all spinors are "pseudo-Majorana" i.e. they satisfy the usual Majorana condition

$$\bar{\xi} = \xi^T C \tag{3.1}$$

but $C^{-1}\Gamma^{\mu}C = +\Gamma^{\mu\ T}$, rather than the usual $C^{-1}\Gamma^{\mu}C = -\Gamma^{\mu\ T}$ of $d = 4$. There exists a basis for the Γ-matrices in which they are all real and in this basis all spinors are also real. We make use of the "unconventional convention" that complex conjugation does not change the order of products of anticommuting spinors, so that no factors of i are required for reality of the Lagrangian.

The fields of the $d = 9$ supergravity multiplet are

$$\{ e_\mu{}^m \ ; \ \psi_\mu \ ; \ A_{\mu\nu\rho\sigma\lambda} \ , \ A_\mu \ ; \ \chi \ ; \ \sigma \}$$

The spin connection $\omega_{\mu mn}$ of the $d = 9$ spacetime is given as usual in terms of the vielbein e_μ^m and its inverse e_m^μ. The Rieman tensor $R_{\mu\nu mn}$ of the $d = 9$ spacetime is defined as

$$R_{\mu\nu mn} = (\partial_\mu \omega_{\nu mn} + \omega_{\mu m}{}^p \omega_{\nu pn}) - \mu \leftrightarrow \nu$$

and the Ricci tensor is given by the contraction $R_{\mu\nu} \equiv R_{\mu\nu mn} e^{\nu n}$. The fields $A_{\mu\nu\rho\sigma\lambda}$ and A_{μ} are Abelian gauge fields with field strengths $G_{\mu\nu\rho\sigma\lambda\eta}$ and $F_{\mu\nu}$, respectively.

The fields of the $d = 9$ Maxwell multiplet are

$$\left\{ A_{\mu} \quad ; \quad \lambda \quad ; \quad \phi \right\}$$

and the combined fields of the $d = 9$ M-E supergravity theory are

$$\left\{ e_{\mu}^{\ m} \ ; \ \psi_{\mu} \ ; \ A_{\mu\nu\rho\sigma\lambda} \ , \ A_{\mu}^{\ I} \ ; \ \lambda^{a} \ , \ \chi \ ; \ \phi^{x} \ , \ \sigma \right\}$$

where $I = 0,1,2,..n$ so that A_{μ}^{I} includes the n vector fields of the Maxwell multiplets and the single vector field of the graviton multiplet. The index a takes the values $1,2,..n$ and is an index of SO(n), the tangent space group of the manifold M' parametrized by the n scalar fields ϕ^{x}, $x = 1,2...n$.

As in previous constructions we introduce a vielbein f_{x}^{a} and its inverse f_{a}^{x} for M', in terms of which the metric is

$$g_{xy} = f_{x}^{\ a} f_{y}^{\ b} \delta_{ab} \equiv f_{x}^{\ a} f_{y}^{\ a} . \tag{3.2}$$

We introduce the connection $\Omega_{x\ ab}$ of M' through the equation

$$f^{a}_{[x\ ,y]} + \Omega_{[y}{}^{a}{}_{b}\, f^{b}_{x]} = 0 \tag{3.3}$$

which determines it implicitly as a function of f_{x}^{a}. This connection is used to define the composite SO(n) gauge potential

$$\mathcal{A}_\mu{}^{ab} \equiv \Omega_x{}^{ab}(\phi)\,\partial_\mu \phi^x \qquad (3.4)$$

and this in turn to define the covariant derivative $\mathfrak{D}_\mu$ of λ^a:

$$(\mathfrak{D}_\mu \lambda)^a = D_\mu \lambda^a + \mathcal{A}_\mu{}^{ab}(\phi)\,\lambda^b, \qquad (3.5)$$

where $D_\mu\lambda^a = \partial_\mu\lambda^a + \frac{1}{4}\omega_\mu{}^{mn}\Gamma_{mn}\lambda^a$ is the ordinary $d = 9$ spacetime covariant derivative acting on spinors.

Finally, we note that $e = \det e_\mu^m$, $R = R_{\mu m}e^{\mu m}$, and that

$$\Sigma \equiv \exp\left(\frac{1}{\sqrt{7}}\sigma\right). \qquad (3.6)$$

With these conventions the $d = 9$ M–E supergravity Lagrangian is

$$\begin{aligned}
e^{-1}\mathcal{L} = & -\tfrac{1}{2}R + \tfrac{1}{2}\bar{\Psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \Psi_\rho - \frac{1}{2\times 6!}\Sigma^{-4} G_{\mu\nu\rho\sigma\lambda\eta}G^{\mu\nu\rho\sigma\lambda\eta} \\
& - \tfrac{1}{4}\Sigma^2 \mathring{a}_{IJ} F_{\mu\nu}^I F^{\mu\nu J} + \tfrac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) + \tfrac{1}{2} g_{xy}(\partial_\mu\phi^x)(\partial^\mu\phi^y) \\
& - \tfrac{1}{2}\bar{\chi}\,\not{\mathfrak{D}}\chi - \tfrac{1}{2}\bar{\lambda}^a(\not{\mathfrak{D}}\lambda)^a + \tfrac{1}{2}(\bar{\chi}\Gamma^\mu\Gamma^\nu\Psi_\mu)\partial_\nu\sigma + \tfrac{1}{2} f_x^a(\bar{\lambda}^a\Gamma^\mu\Gamma^\nu\Psi_\mu)\partial_\nu\phi^x \\
& + \tfrac{1}{4}\Sigma h_I^a F_{\rho\sigma}^I(\bar{\lambda}^a\Gamma^\mu\Gamma^{\rho\sigma}\Psi_\mu) + \frac{1}{4\sqrt{7}}\Sigma h_I F_{\rho\sigma}^I(\bar{\chi}\Gamma^\mu\Gamma^{\rho\sigma}\Psi_\mu) \\
& - \frac{5}{56}\Sigma h_I F_{\rho\sigma}^I(\bar{\chi}\Gamma^{\rho\sigma}\chi) + \frac{1}{8}\Sigma h_I F_{\rho\sigma}^I(\bar{\lambda}^a\Gamma^{\rho\sigma}\lambda^a) + \frac{1}{2\sqrt{7}}\Sigma h_I^a F_{\rho\sigma}^I(\bar{\lambda}^a\Gamma^{\rho\sigma}\chi) \\
& + \tfrac{1}{8}\Sigma F_{\rho\sigma}^I h_I\left[\bar{\Psi}_\mu(\Gamma^{\mu\nu\rho\sigma} + 2g^{\mu\rho}g^{\nu\sigma})\Psi_\nu\right] + \frac{1}{\sqrt{14}\times 6!}\Sigma^{-2} G_{\mu\nu\rho\sigma\lambda\kappa}(\bar{\chi}\Gamma^\delta\Gamma^{\mu\nu\rho\sigma\lambda\kappa}\Psi_\delta) \\
& - \frac{\sqrt{2}}{8\times 7!}\Sigma^{-2} G_{\mu\nu\rho\sigma\lambda\kappa}(\bar{\chi}\Gamma^{\mu\nu\rho\sigma\lambda\kappa}\chi) + \frac{\sqrt{2}}{8\times 6!}\Sigma^{-2} G_{\mu\nu\rho\sigma\lambda\kappa}(\bar{\lambda}^a\Gamma^{\mu\nu\rho\sigma\lambda\kappa}\lambda^a) \\
& + \frac{\sqrt{2}}{\cdots}\Sigma^{-2} G_{\mu\nu\rho\sigma\lambda\kappa}\left[\bar{\Psi}_\delta(\Gamma^{\delta\gamma\mu\nu\rho\sigma\lambda\kappa} + 30 g^{\delta\mu}g^{\gamma\nu}\Gamma^{\rho\sigma\lambda\kappa})\Psi_\gamma\right] +
\end{aligned}$$

$$+ \frac{1}{4\sqrt{2}\times 5!} C_{IJ}\, e^{-1} \varepsilon^{\mu\nu\rho\sigma\lambda\kappa\delta\gamma\tau} F^I_{\mu\nu} F^J_{\rho\sigma} A_{\lambda\kappa\delta\gamma\tau} + \mathcal{L}_4 , \tag{3.7}$$

where L_4 are the 4-fermion terms, which we have not computed. The supersymmetry transformaton laws are

$$\delta e_\mu{}^m = \bar\epsilon \Gamma^m \psi_\mu \quad , \quad \delta\sigma = \tfrac{1}{2}\bar\epsilon\chi \quad , \quad \delta\phi^x = \tfrac{1}{2} f_a^x\, \bar\epsilon\lambda^a$$

$$\delta\psi_\mu = D_\mu\epsilon + \frac{1}{28}\Sigma h_I F^I_{\rho\sigma}\left(\Gamma_\mu{}^{\rho\sigma} - 12\delta_\mu^\rho\Gamma^\sigma\right)\epsilon - \frac{5\sqrt{2}}{4\times 7!} G_{\nu\rho\sigma\lambda\kappa\delta}\left(\Gamma_\mu{}^{\nu\rho\sigma\lambda\kappa\delta} - \frac{12}{5}\delta_\mu^\nu\Gamma^{\rho\sigma\lambda\kappa\delta}\right)\epsilon\,\Sigma^{-2} + \delta_3$$

$$\delta A^I_\mu = -\tfrac{1}{2}\Sigma^{-1} h^I_a\, \bar\epsilon\Gamma_\mu\lambda^a - \frac{1}{2\sqrt{7}}\Sigma^{-1} h^I\, \bar\epsilon\Gamma_\mu\chi + \tfrac{1}{2}\Sigma^{-1}\bar\epsilon\psi_\mu h^I$$

$$\delta A_{\mu\nu\rho\sigma\lambda} = -\frac{1}{\sqrt{14}}\Sigma^2\,\bar\epsilon\Gamma_{\mu\nu\rho\sigma\lambda}\chi - \frac{5}{2\sqrt{2}}\Sigma^2\,\bar\epsilon\Gamma_{[\mu\nu\rho\sigma}\psi_{\lambda]}$$

$$\delta\chi = \tfrac{1}{2}(\not\partial\sigma)\epsilon + \frac{1}{4\sqrt{7}}\Sigma h_I F^I_{\rho\sigma}\Gamma^{\rho\sigma}\epsilon + \frac{1}{\sqrt{14}\times 6!}\Sigma^{-2} G_{\mu\nu\rho\sigma\lambda\kappa}\Gamma^{\mu\nu\rho\sigma\lambda\kappa}\epsilon + \delta_3$$

$$\delta\lambda^a = \tfrac{1}{2} f_x^a(\not\partial\phi^x)\epsilon + \tfrac{1}{4}\Sigma h^a_I F_{\rho\sigma}\Gamma^{\rho\sigma}\epsilon + \delta_3 , \tag{3.8}$$

where δ_3 are 3-fermion terms that are needed only for the computation of L_4

The quantities h_I, h^I, h^a_I, h^{Ia} and $\mathring{a}_{IJ}$ are functions of the scalar field ϕ^x, whereas C_{IJ} is constant, as required by gauge invariance of the action. These quantities are subject to algebraic and differential constraints. The algebraic constraints are

$$\mathring{a}_{IJ} = h_I h_J + h^a_I h^a_J \tag{3.9a}$$

$$C_{IJ} = 2 h_I h_J - \mathring{a}_{IJ} \tag{3.9b}$$

and

$$h^I h_I = 1$$
$$h^{Ia} h_I^b = \delta^{ab}$$
$$h^I h_{Ia} = h_I h^{Ia} = 0 \tag{3.10}$$

It follows from (3.9a) and (3.10) that

$$h_I = \mathring{a}_{IJ} h^J$$
$$h_I^a = \mathring{a}_{IJ} h^{Ja} \tag{3.11}$$

i.e. that $\mathring{a}_{IJ}$ serves as a "metric" for I, J indices.

The differential constraints are

$$h_{I,x} = f_x^a h_I^a$$
$$h^I{}_{,x} = -f_x^a h^{Ia} \tag{3.12}$$

and

$$h_I^a{}_{;x} = f_x^a h_I$$
$$h^{Ia}{}_{;x} = -f_x^a h^I \tag{3.13}$$

where the semi-colon indicates covariant differentiation on $\mathcal{M}'$. In principle the algebraic constraint (3.9b) implies an additional differential constraint following from $C_{IJ,x} = 0$, but this is identically satisfied as a consequence of (3.12) and (3.13). We remark also that

$$C^{IJ} \equiv a^{II'} a^{JJ'} C_{I'J'} \tag{3.14}$$

where a^{IJ} are the components of a^{-1}, is also <u>constant</u>, i.e.

$$C^{IJ}{}_{,x} = 0 \tag{3.15}$$

despite the fact that a is a function of ϕ. (This fact has a direct analogue in $d = 5$ when the model is one of those associated with a Jordan algebra [1]).

The differential constraints (3.12) and (3.13) have as their integrability conditions the constraint

$$K_{xyuv} = -2\, g_{u[x}\, g_{y]v} \tag{3.16}$$

on the Riemann tensor K of M', in agreement with ref(5), and with (2.37).

Now the constraints (3.9) and (3.10) imply that

$$\mathcal{N}(h) = C_{IJ}\, h^I h^J = 1 \tag{3.17}$$

so that, referring to the discussion of the previous section, we see that the functions h^I are interpretable as the coordinates of a cone C restricted to the $\mathcal{N} = 1$ hypersurface, with $\ln \mathcal{N}$ being the kernel of the natural metric a_{IJ} on C. The function $\mathring{a}_{IJ}$ is simply the restriction of a_{IJ} to this hypersurface, which is to be identified with the manifold $M' = SO(n,1)/SO(n)$ whose coordinates are the scalar fields ϕ^x. As explained in the previous section the quadratic form $\mathcal{N}$ can also be interpreted as the norm form of a Jordan algebra of degree two. The supergravity model is entirely determined by this associated algebra as all the scalar field interactions follow from a knowledge of $\mathcal{N}$. We shall now show how the Lagrangian (3.7) can be rewritten in such a way as to make this fact evident. To this end we define

$$\tilde{h}_I = \Sigma^{\sqrt{7}} h_I$$

$$\tilde{h}^I = \Sigma^{-\sqrt{7}} h^I$$

$$a_{IJ} = \Sigma^{2\sqrt{7}} \mathring{a}_{IJ} \tag{3.18}$$

The kinetic terms for the scalar fields ϕ^x <u>and</u> σ, can now be rewritten jointly as

$$\frac{1}{2} a_{IJ} (\partial_\mu \tilde{h}^I)(\partial^\mu \tilde{h}^J) \tag{3.19}$$

so that the $\tilde{h}^I$ may now be considered as $n + 1$ independent scalar fields which parametrize a space with metric a_{IJ}. Whereas previously we had $N(h) = 1$, so that the space M', parametrized by the ϕ^x, was the $N = 1$ hypersurface of the cone C, now we have

$$\mathcal{N}(\tilde{h}) = C_{IJ} \tilde{h}^I \tilde{h}^J = \Sigma^{-2\sqrt{7}} = e^{-2\sigma} \tag{3.20}$$

so that the variables $\tilde{h}^I$ may be considered as coordinates for the <u>entire</u> cone C. Clearly a_{IJ} is the metric of C, as the notation suggests, and from the previous section we know that it is derivable from $\mathcal{N}$ as (ν = 2)

$$a_{IJ} = -\frac{1}{2} \partial_{IJ} \ln \mathcal{N}(\tilde{h}) \tag{3.21}$$

(where $\partial_I = \partial/\partial\tilde{h}^I$). Indeed, making the identification

$$\tilde{h}_I = a_{IJ}\tilde{h}^J = \frac{1}{2}\partial_I \ln \mathcal{N}(\tilde{h}) \tag{3.22}$$

as required by the interpretation of $\tilde{h}^I$ as the coordinates of a point in C, it is straightforward to verify that

$$a_{IJ} = -\frac{1}{\mathcal{N}} C_{IJ} + 2\tilde{h}_I\tilde{h}_J \tag{3.23}$$

which is equivalent to (3.9b). We remark, further, that (3.20) is equivalent to

$$\sigma = -\frac{1}{2} \ln \mathcal{N}(\tilde{h}) \tag{3.24}$$

so that the scalar field σ when written as a function of the coordinates $\tilde{h}$ of C is nothing but the kernel of the natural metric on C !

We now proceed to rewrite the Lagrangian in terms of the variables $\tilde{h}^I$. We define

$$\tilde{h}^I{}_a = \Sigma^{-\sqrt{7}} h^I{}_a \tag{3.25}$$

and we remark that as ∂/∂_σ commutes with $\partial/\partial\phi^x$ the differential constraints relating $\tilde{h}_I$ and $\tilde{h}_I^a$ are the same as those relating h_I and h_I^a.

We shall find it convenient to define

$$\tilde{h}_I{}^A = \left(\tilde{h}_I \; , \; \tilde{h}_I{}^a \right)$$
$$\tilde{T}_I{}^A = \left(h_I \; , \; \sqrt{7}\, h_I^a \right) \tag{3.26}$$

and

$$C^A = \tilde{h}^I \tilde{h}_I{}^A = \delta^{A0} \tag{3.27}$$

and

$$\tilde{T}_I{}^{AB} = -\frac{1}{56}\left[\tilde{h}_I \left(\eta^{AB} + 8C^A C^B \right) - 4\tilde{T}_I{}^{(A} C^{B)} \right] \tag{3.28}$$

where

$$\eta^{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -7\delta_{ab} \end{pmatrix} \tag{3.29}$$

We can now write the multiplet of the d = 9 M–E supergravity as

$$\left\{ e_\mu{}^m ; \; \psi_\mu \; ; \; A_{\mu\nu\rho\lambda} \; , \; A_\mu{}^I \; ; \; \chi^A \; ; \; \tilde{h}^I \right\}$$

in which we have combined the n + 1 spinors λ^a and χ into the single set of spinors χ^A, A = 0,1,2,...n. The covariant derivative of χ^A is

$$(\mathcal{D}_\mu \chi)^A = D_\mu \chi^A + A_\mu^{AB} \chi^B \text{ where}$$

$$A_\mu^{AB} \equiv \begin{pmatrix} 0 & 0 \\ 0 & A_\mu^{ab}(\phi) \end{pmatrix} . \tag{3.30}$$

In this notation the Lagrangian takes the form

$$\begin{aligned}
e^{-1}\mathcal{L} = & -\frac{1}{2}R + \frac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{2\times 6!}\Sigma^{-4} G_{\mu\nu\rho\sigma\lambda\kappa} G^{\mu\nu\rho\sigma\lambda\kappa} \\
& -\frac{1}{4}\Sigma^{2\alpha} a_{IJ} F^I_{\mu\nu} F^{\mu\nu J} + \frac{1}{2} a_{IJ}(\partial_\mu \tilde{h}^I)(\partial^\mu \tilde{h}^J) - \frac{1}{2}\bar{\chi}^A (\not{\mathcal{D}}\chi)^A \\
& + \frac{1}{2}\bar{\chi}^A \Gamma^\mu \Gamma^\nu \psi_\mu (\tilde{h}^I \partial_\nu \tilde{h}_I^A) + \frac{1}{4\sqrt{7}}\Sigma^\alpha \tilde{T}_I^A F^I_{\rho\sigma}(\bar{\chi}^A \Gamma^\mu \Gamma^{\rho\sigma}\psi_\mu) \\
& + \Sigma^\alpha \tilde{T}_I^{AB} F^I_{\rho\sigma}(\bar{\chi}^A \Gamma^{\rho\sigma}\chi^B) + \frac{1}{8}\Sigma^\alpha \tilde{h}_I F^I_{\rho\sigma}\left[\bar{\psi}_\mu(\Gamma^{\mu\nu\rho\sigma} + 2\eta^{\mu\rho}\eta^{\nu\sigma})\psi_\nu\right] \\
& + \frac{1}{\sqrt{14}\times 6!}\Sigma^{-2} G_{\mu\nu\rho\sigma\lambda\kappa} C^A (\bar{\chi}^A \Gamma^\tau \Gamma^{\mu\nu\rho\sigma\lambda\kappa}\psi_\tau) \\
& -\frac{\sqrt{2}}{8}\Sigma^{-2} G_{\mu\nu\rho\sigma\lambda\kappa}\eta^{AB}(\bar{\chi}^A \Gamma^{\mu\nu\rho\sigma\lambda\kappa}\chi^B) \\
& + \frac{7\sqrt{2}}{8}\Sigma^{-2} G_{\mu\nu\rho\sigma\lambda\kappa}\left[\bar{\psi}_\delta(\Gamma^{\delta\tau\mu\nu\rho\sigma\lambda\kappa} + 30\eta^{\delta\mu}\eta^{\tau\nu}\Gamma^{\rho\sigma\lambda\kappa})\psi_\tau\right] \\
& + \frac{e^{-1}}{4\sqrt{2}\times 5!} C_{IJ}\,\varepsilon^{\mu\nu\rho\sigma\lambda\kappa\delta\gamma\tau} F^I_{\mu\nu} F^J_{\rho\sigma} A_{\lambda\kappa\delta\gamma\tau} + \mathcal{L}_4 .
\end{aligned} \tag{3.31}$$

The supersymmetry transformation laws are

$$\delta e_\mu{}^m = \frac{1}{2}\bar{\epsilon}\Gamma^m \psi_\mu , \quad \tilde{h}^I \delta \tilde{h}_I^A = \frac{1}{2}\bar{\epsilon}\chi^A$$

$$\begin{aligned}
\delta\psi_\mu = & D_\mu \epsilon + \frac{1}{28}\Sigma^\alpha \tilde{h}_I F^I_{\rho\sigma}(\Gamma_\mu{}^{\rho\sigma} - 12\delta_\mu{}^\rho \Gamma^\sigma)\epsilon \\
& - \frac{5\sqrt{2}}{4\times 7!}\Sigma^{-2} G_{\nu\rho\sigma\lambda\kappa\delta}\left(\Gamma_\mu{}^{\nu\rho\sigma\lambda\kappa\delta} - \frac{12}{5}\delta_\mu{}^\nu \Gamma^{\rho\sigma\lambda\kappa\delta}\right)\epsilon + \delta_3
\end{aligned}$$

$$\delta A^I_\mu = -\frac{1}{2\sqrt{7}}\Sigma^{-\alpha}\tilde{T}^{IA}\bar{\epsilon}\Gamma_\mu \chi^A + \frac{1}{2}\Sigma^{-\alpha}\bar{\epsilon}\psi_\mu \tilde{h}^I$$

$$\delta A_{\mu\nu\rho\sigma\lambda} = -\frac{1}{\sqrt{14}} C^A \Sigma^2 \bar{\epsilon}\Gamma_{\mu\nu\rho\sigma\lambda}\chi^A - \frac{5}{2\sqrt{2}}\Sigma^2 \bar{\epsilon}\Gamma_{[\mu\nu\rho\sigma}\psi_{\lambda]}$$

$$\delta\chi^{A} = \frac{1}{2}\tilde{h}^{I}\not{\partial} h_{I}^{A}\,\epsilon + \frac{1}{4\sqrt{7}}\Sigma^{\alpha}\,\tilde{T}_{I}^{A}\,F_{\rho\sigma}^{I}\,\Gamma^{\rho\sigma}\epsilon + \frac{1}{\sqrt{14}\times 6!}\Sigma^{-1}c^{A}\,G_{\mu\nu\rho\sigma\lambda\kappa}\,\Gamma^{\mu\nu\rho\sigma\lambda\kappa}\,\epsilon \tag{3.32}$$

Here, the number α is

$$\alpha = 1 - \sqrt{7}$$

Our claim that the full scalar manifold M is the cone C of a Jordan algebra of degree 2 is now manifest. Notice also that the coefficient of the kinetic term of the vector fields is simply a multiple of the metric of C, so that this construction provides a geometrical interpretation of this function, as in the N = 2 d = 5 case.

4. $S\ell(2;R)$ gauging.

For $n = 2$ the non-compact symmetry group of the $d = 9$ M-E theory is $SO(2,1) = S\ell(2;R)$. As discussed in the introduction this group may be gauged by the three gauge fields A_μ^I without destroying the irreducibility of the ungauged theory. In this section we shall construct this model.

Firstly we need the explicit form of the $S\ell(2;R)$ transformations. Those that are non-zero are

$$\delta \phi^x = K_I^x(\phi)\, \alpha^I$$

$$\delta \lambda^a = L_I^{ab}(\phi)\, \lambda^b \alpha^I$$

$$\delta A_\mu^I = (M_K)_J{}^I A_\mu^J \alpha^K \tag{4.1}$$

where α^k is an infinitesimal parameter. As the 3 rep. of $Sl(2;R)$ is the adjoint rep. the matrices M_k are simply

$$(M_K)_J{}^I = f_{KJ}{}^I \tag{4.2}$$

where the $f_{KJ}{}^I$ are the $S\ell(2;R)$ structure constants. The quantities K_I^x are the Killing vectors of $S\ell(2;R)/SO(2)$ that generate the $S\ell(2;R)$ isometry group, and

$$L_I^{ab}(\phi) = K_I^{a;b} - \Omega_Y{}^{ab} K_I{}^Y \tag{4.3}$$

with $K_I^a = K_I^x f_x^a$. These Killing vectors can be expressed in terms of the structure constants of $S\ell(2;R)$ as

$$K_I{}^a = f_{IJ}{}^K h_K(\phi) h^{Ja}(\phi) \tag{4.4}$$

The rigid $S\ell(2;R)$ invariance is now promoted to a local gauge invariance by covariantisation of the action. We make the following replacements

$$\partial_\mu \phi^x \rightarrow (\hat{\partial}_\mu \phi)^x = \partial_\mu \phi^x + g A_\mu^I K_I^x$$

$$(\mathcal{D}_\mu \lambda)^a \rightarrow (\hat{\mathcal{D}}_\mu \lambda)^a = (\mathcal{D}_\mu \lambda)^a + g A_\mu^I L_I{}^{ab}(\phi)\lambda^b$$

$$F_{\mu\nu}^I \rightarrow \hat{F}_{\mu\nu}^I = F_{\mu\nu}^I + g f_{KJ}{}^I A_\mu{}^J A_\nu{}^K \tag{4.5}$$

where g is the $S\ell(2;R)$ gauge coupling constant. The action is now invariant under the gauge transformations

$$\delta\phi^x = K_I^x \, \alpha^I(x)$$

$$\delta\lambda^a = L_I{}^{ab}(\phi)\, \alpha^I(x)$$

$$\delta A_\mu^I = -\frac{1}{g}\left(\hat{\partial}_\mu \alpha(x)\right)^I \equiv -\frac{1}{g}\partial_\mu \alpha^I + f_{KJ}{}^I A_\mu^J \alpha^K \tag{4.6}$$

This covariantization breaks supersymmetry which is restored by the addition to the Lagrangian $\mathcal{L}$ of the term

$$\mathcal{L}' = -\frac{g}{2} h^I_{[a} K_{I\, b]} \left(\bar{\lambda}^a \lambda^b\right) \tag{4.7}$$

The supersymmetry transformation rules need no modification. Notice in particular that no scalar potential is required. This is to be expected from the fact that the gravitino is $S\ell(2;R)$ inert.

These results are very similar to those of the SU(3,1) gauging of N = 2 d = 5 M-E supergravity [11], the principal difference being that here naive covariantization is sufficient, no Chern-Simon terms being required.

As $S\ell(2;R)$ is non-linearly realised (as must be the case for a non-compact gauge group if ghosts are to be avoided) it is not a symmetry of the ground state. It is broken to U(1), its maximally compact subgroup. The gauge fields A^I_μ decompose into a singlet and a doublet (i.e. a pair of fields of charges +1 and -1) of U(1). The singlet can be identified as the gauge field in the graviton multiplet and it gauges the unbroken U(1) factor. The remaining charged pair of gauge fields acquire a mass, as do the spinors λ^a (from the new term (4.7) in the Lagrangian). These massive fields form a massive centrally charged d = 9 vector multiplet, the central charge being the U(1) charge. Thus we have effectively constructed a model with a gauged central charge. Again this is all directly analogous to the N = 2 d = 5 SU(3,1) gauging.

5. Remarks.

As we remarked in the introduction, the d = 9 M-E theory can be dimensionally reduced to give a class of d = 8 M-E theories. If one takes the Lagrangian in the form (3.7), where ϕ^x and σ are the independent fields the reduction proceeds in exactly the same way as for d = 5 d = 4, and with the same result, viz. that $M' = Str_0(J)/Aut(J)$ is replaced by the Kählerian manifold $Mö(J)/Str(J)$ parametrized by $(n+1)$ complex fields. The scalar field σ plays no part in this reduction but simply carries over to d = 8 as the scalar field of the d = 8 supergravity multiplet. The scalar field manifolds in d = 8 obtained by dimensional reduction from d = 9 are therefore locally of the form

$$SO(1,1) \times SO(m,2)/[SO(m) \times SO(2)] \quad ; \quad m \geq 1$$

But this is also the general result [8], so all $d = 8$ M-E theories have a $d = 9$ counterpart.

It seems very likely that this is true for $d < 8$. In this case all maximal M-E supergravity theories in $d \leqslant 9$ must be determined by the quadratic polynomial N. For $d = 8$ this comes about because

$$K(z,\bar{z}) = \ln N(z+\bar{z})$$

is the kernel for the Kähler metric on $SO(m,2)/[SO(m)\times SO(2)]$.
Note that the real ranks of the symmetric spaces $SO(n,1)/SO(n)$ and $SO(m,2)/[SO(m)\times SO(2)], m>2$, are one and two respectively. The increase by one in the real rank of the scalar field manifold under dimensional reduction on a torus is a general feature of supergravity theories. Hence further dimensional reduction to d-dimensional spacetime will result in a supergravity theory with scalar field manifold M_d of real rank 10-d. Specifically

$$M_d = \frac{SO(n+k,\,10-d)}{SO(n+k)\times SO(10-d)}, \quad k = 9-d$$

In d=7 the manifold M_d is quaternionic for n=1 and in d=6 it is quaternionic for all n. In d=2 the manifold has real rank 8 and may be parametrized by octonions. It would be very interesting to know whether the scalar manifolds in dimensions d<8 can also be described by a " Kernel function" which is a hypercomplex (quaternionic etc.) extension of the norm function in d=9.

Finally, one interesting facet of the Lagrangian (3.31) is that the spin 3/2 and spin 1/2 kinetic terms are of opposite sign. We agree on this point with ref. (4), and it is also true of the d=8 M-E supergravity theory. We do not beleive that this is a matter of convention but we have no explanation for it. Unlike the signs of bosonic kinetic terms those of fermionic kinetic terms are not fixed by positivity of the energy, so there is no priori reason for the latter to have any particular sign.

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